Subgraph Counting in Practice

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July 29, 2019

What is the subgraph counting problem?

Problem Statement

How many (unlabelled) copies of the graph H are contained in the graph G?

We call the graph G the *host* graph and H the *pattern* graph.

Algorithmic complexity

- Subgraph isomorphism is NP-complete $\rightarrow$ subgraph counting is NP-hard

Fixed-parameter tractable (FPT)

A problem is FPT with respect to parameter k if it can be solved by an algorithm with runtime $f(k) \times n^{O(1)}$

- Assuming Exponential Time Hypothesis $\rightarrow$ subgraph counting is not FPT in general

Algorithmic complexity

Almost bounded degree graph

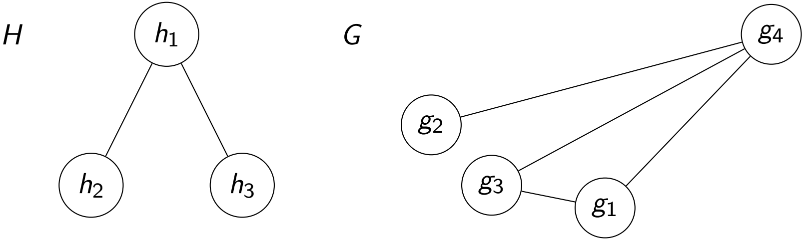
A graph G has *almost bounded degree* k if G contains at most k vertices with degree greater than k.

- (Enright and Meeks) Subgraph counting is FPT for host graphs with *almost bounded degree* and small pattern graphs

FPT Algorithm: General Idea

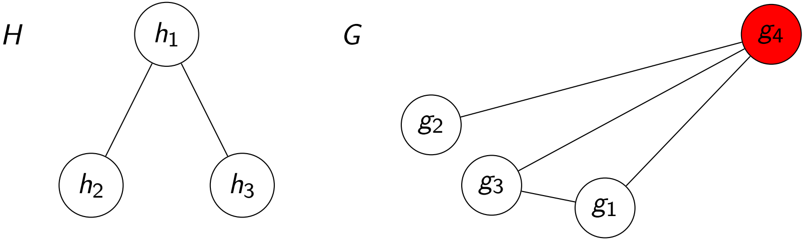
- Consider each way to assign part of H to the k high degree vertices in G
- For each feasible assignment, count ways to assign the rest of H to the bounded degree part of G
- Sum up the counts to obtain number of labelled copies of H in G
- Divide by the number of symmetries in H to obtain number of *unlabelled* copies of H in G

Example



$h_1 : g_1, g_2, g_3, g_4$
$h_2 : g_1, g_2, g_3, g_4$
$h_3 : g_1, g_2, g_3, g_4$

Example



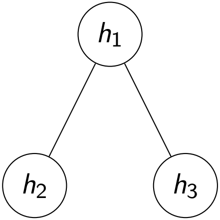
$h_1 : g_1, g_2, g_3, g_4$

$h_2 : g_1, g_2, g_3, g_4$

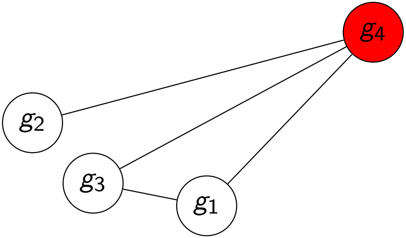
$h_3 : g_1, g_2, g_3, g_4$

Example

H



G

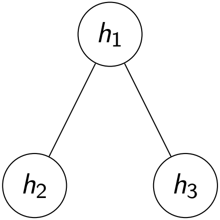


Subset of $V(H)$: $\emptyset$
Count $= 0$

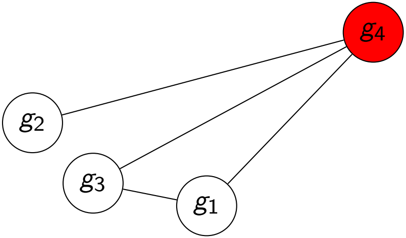
$h_1: g_1, g_2, g_3, \cancel{g_4}$
$h_2: g_1, g_2, g_3, \cancel{g_4}$
$h_3: g_1, g_2, g_3, \cancel{g_4}$

Example

H



G



Subset of $V(H)$: $\emptyset$
Count $= 0$

$h_1 \to g_1$
h_2 : ~~g_1~~, ~~g_2~~, g_3
h_3 : ~~g_1~~, ~~g_2~~, g_3

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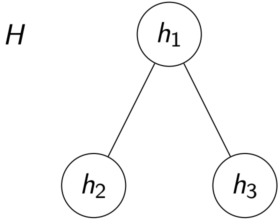
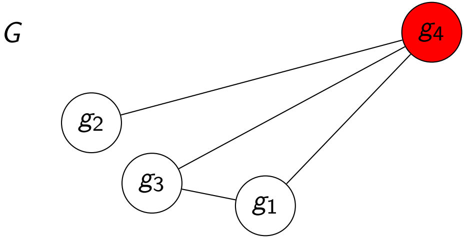
Example

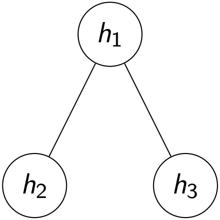




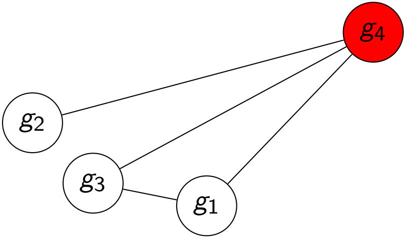
Subset of $V(H)$: $\emptyset$
Count $= 0$

$h_1 \rightarrow g_2$
$h_2 : \cancel{g_1}, \cancel{g_2}, \cancel{g_3}$
$h_3 : \cancel{g_1}, \cancel{g_2}, \cancel{g_3}$

H



G



Subset of $V(H)$: $\emptyset$
Count $= 0$

$h_1 \to g_3$
$h_2 : g_1, \cancel{g_2}, \cancel{g_3}$
$h_3 : g_1, \cancel{g_2}, \cancel{g_3}$

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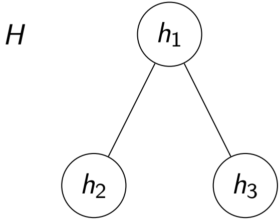
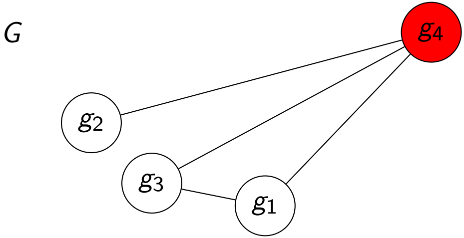
Example





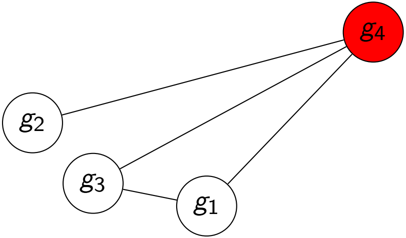
Subset of $V(H)$: h_1
Count $= 0$

$h_1 \to g_4$
$h_2 : g_1, g_2, g_3$
$h_3 : g_1, g_2, g_3$

Example

$H \setminus h_1$

G



Connected components of $H \setminus h_1$:
$C_1 = h_2$
$C_2 = h_3$

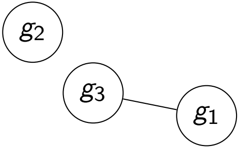
Example

$$\begin{aligned}\text{copies of } C_1 \text{ and } C_2 \text{ in } G \setminus g_4 = {} & (\text{copies of } C_1 \text{ in } G \setminus g_4 \\ & \times \text{copies of } C_2 \text{ in } G \setminus g_4) \\ & - \text{overlapping copies of } C_1 \text{ and } C_2 \text{ in } G \setminus g_4\end{aligned}$$

Counting copies of C_1 in $G \setminus g_4$

C_1

$G \setminus g_4$



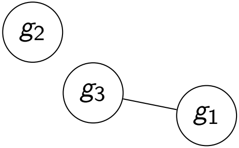
Count $=0$

$h_2: g_1, g_2, g_3$

Counting copies of C_1 in $G \setminus g_4$

C_1

$G \setminus g_4$

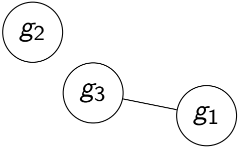


Count $= 0 + 3$

Counting copies of C_2 in $G \setminus g_4$

C_2

$G \setminus g_4$



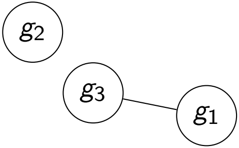
Count $=0$

$h_3 : g_1, g_2, g_3$

Counting copies of C_2 in $G \setminus g_4$

C_2

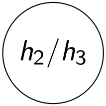
$G \setminus g_4$



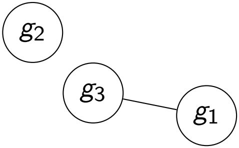
Count $= 0 + 3$

Counting overlapping copies of C_1 and C_2 in $G \setminus g_4$

$C_1 \cap C_2$



$G \setminus g_4$

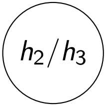


Count $=0$

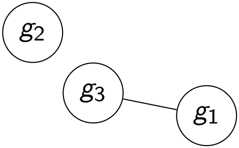
h_2/h_3: g_1, g_2, g_3

Counting overlapping copies of C_1 and C_2 in $G \setminus g_4$

$C_1 \cap C_2$



$G \setminus g_4$



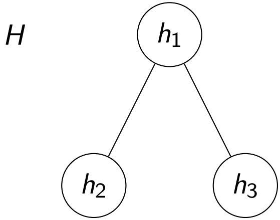
Count $= 0 + 3$

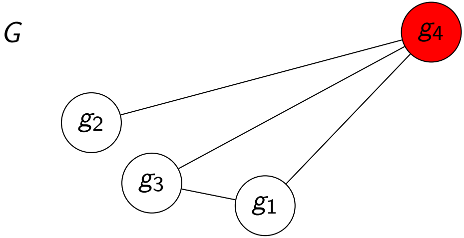
$h_2/h_3 \rightarrow g_1$

Example

$$\text{copies of } C_1 \text{ and } C_2 \text{ in } G \setminus g_4 = (3 \times 3) - 3$$
$$= 6$$

Example

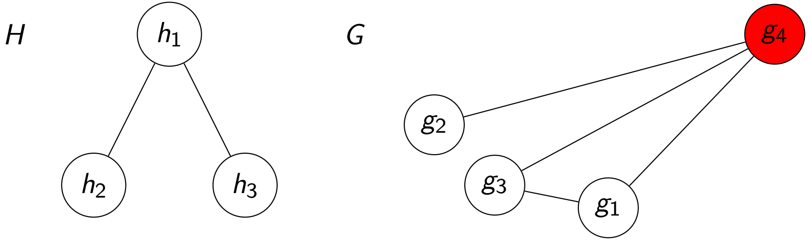




Subset of $V(H)$: h_1

Count $= 0+6$

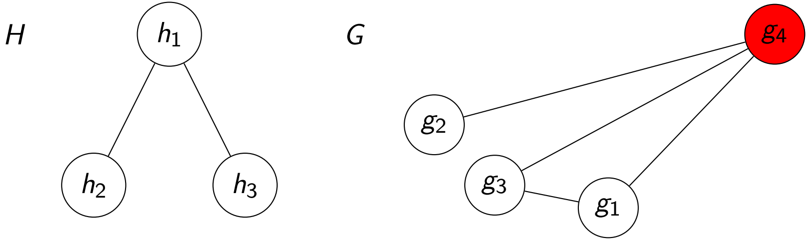
Example



Subset of $V(H)$: h_2
Count $= 6$

h_1 : $g_1, g_2, g_3,$ ~~g_4~~
h_2 : ~~g_1~~, ~~g_2~~, ~~g_3~~, g_4
h_3 : $g_1, g_2, g_3,$ ~~g_4~~

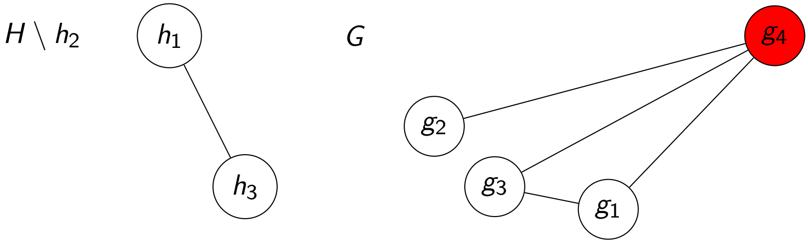
Example



Subset of $V(H)$: h_2
Count $= 6$

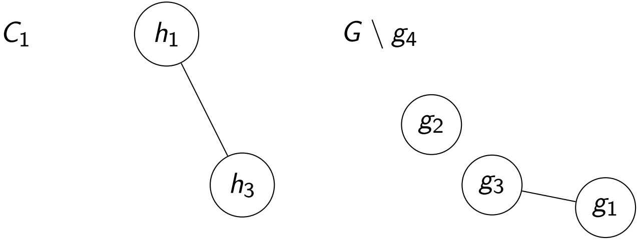
$h_1: g_1, g_2, g_3$
$h_2 \rightarrow g_4$
$h_3: g_1, g_2, g_3$

Example



Connected components of $H \setminus h_2$:
$C_1 = h_1, h_3$

Counting copies of C_1 in $G \setminus g_4$

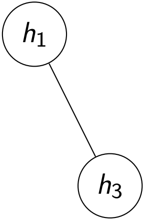


Count $=0$

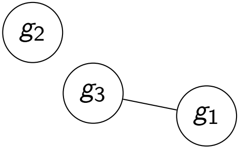
$h_1 : g_1, g_2, g_3$
$h_3 : g_1, g_2, g_3$

Counting copies of C_1 in $G \setminus g_4$

C_1



$G \setminus g_4$

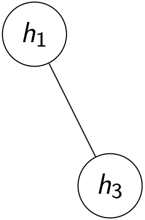


Count $=0$

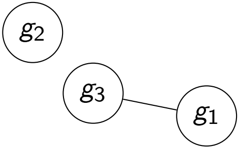
$h_1 \rightarrow g_1$
h_3 : ~~g_1~~, ~~g_2~~, g_3

Counting copies of C_1 in $G \setminus g_4$

C_1



$G \setminus g_4$

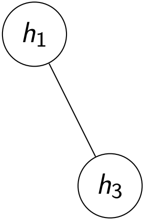


Count $=0+1$

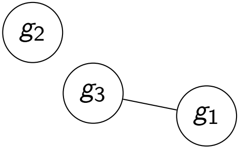
$h_1 \rightarrow g_1$
$h_3 \rightarrow g_3$

Counting copies of C_1 in $G \setminus g_4$

C_1



$G \setminus g_4$

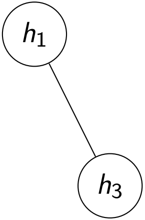


Count $=1$

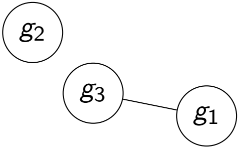
$h_1 \rightarrow g_2$
h_3 : ~~g_1~~, ~~g_2~~, ~~g_3~~

Counting copies of C_1 in $G \setminus g_4$

C_1



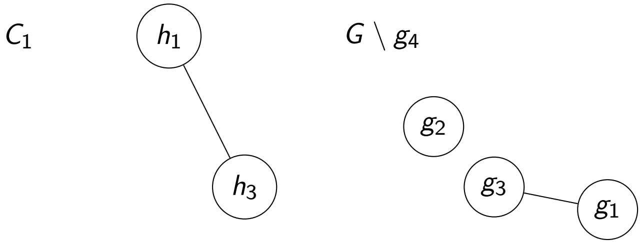
$G \setminus g_4$



Count $=1$

$h_1 \rightarrow g_3$
h_3 : g_1, ~~g_2~~, ~~g_3~~

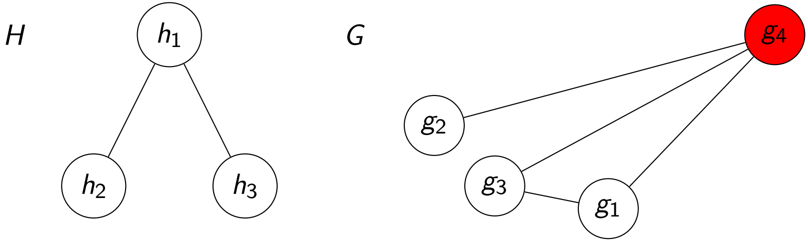
Counting copies of C_1 in $G \setminus g_4$



Count $= 1 + 1$

$h_1 \rightarrow g_3$
$h_3 \rightarrow g_1$

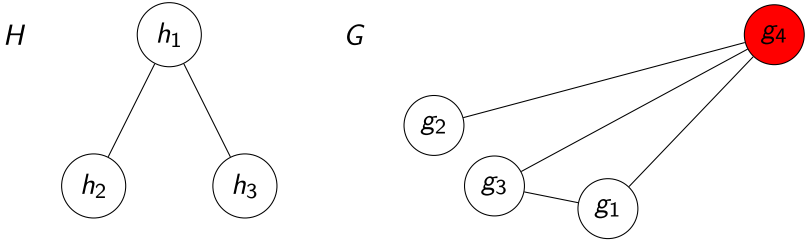
Example



Subset of $V(H)$: h_2

Count $= 6+2$

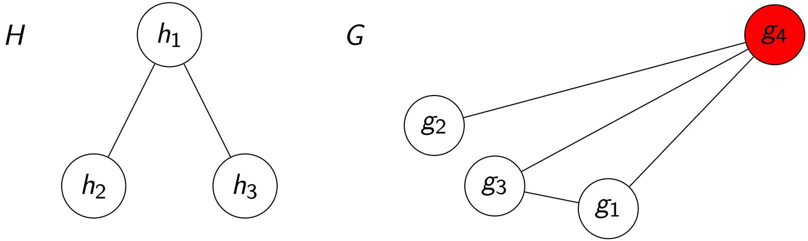
Example



Subset of $V(H)$: h_3

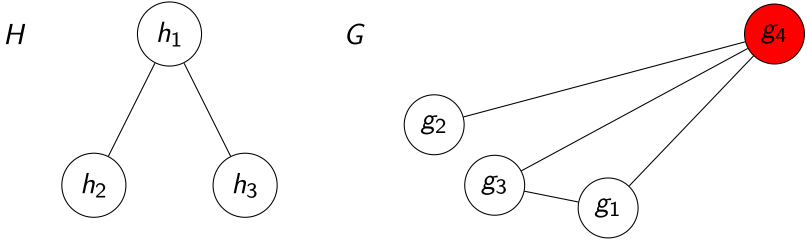
Count $= 8+2$

Example



Count $= 10$

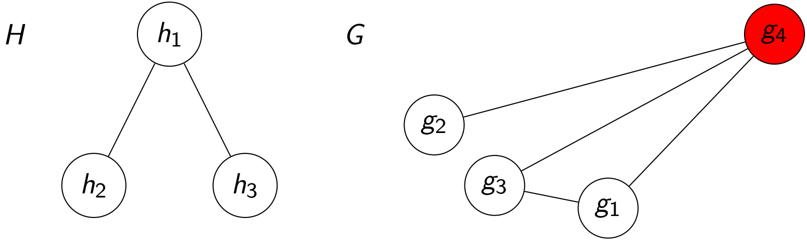
Example



Count $= 10$

Number of copies of H in $H = 2$

Example



Count $= 10$

Number of copies of H in $H = 2$

$\rightarrow$ Number of unlabelled copies of H in $G = 10/2 = 5$

Thank You!